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Approximation by Multiinteger Translates of Functions Having Global Support

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We consider L_p-approximation $(1 \leqslant p \leqslant \infty)$ by multiinteger translates of several functions which are not necessarily compactly supported but have a suitable decay rate. In particular, we introduce a modified notion of controlled approximation and characterize the controlled approximation order in terms of the Strang–Fix conditions.

1. INTRODUCTION

In this paper we consider L_p-approximation $(1 \leqslant p \leqslant \infty)$ by the multiinteger translates of a finite number of functions which are not necessarily compactly supported but have a suitable decay rate. In particular, we introduce a modified notion of controlled approximation and characterize the controlled approximation order in terms of the Strang–Fix conditions (see [15]). Our results extend the recent interesting work of Light and Cheney [12].

Integer translates of a function on $\mathbb{R}$ were already considered by Schoenberg in his celebrated paper [13]. Let ϕ be an exponentially decaying function on $\mathbb{R}$. For a sequence c on $\mathbb{Z}$, the semi-discrete convolution product $\phi *' c$ is the function given by

$$\phi *' c = \sum_{j \in \mathbb{Z}} \phi(\cdot - j)\, c(j).$$

We say that the mapping $\phi *'$ preserves polynomials of degree $k-1$, if for any polynomial p of degree $d \leqslant k-1$, $\phi *' p - p$ is a polynomial of degree $< d$. Schoenberg proved that $\phi *'$ preserves polynomials of degree $k-1$ if the Fourier transform $\hat{\phi}$ satisfies the conditions $\hat{\phi}(0) \neq 0$ and $D^{\alpha}\hat{\phi}(2\pi j) = 0$ for $0 \leqslant \alpha < k$ and $j \in \mathbb{Z} \setminus \{0\}$.

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In the finite element method, multivariate piecewise polynomial functions are frequently used. In the 1970s, Strang and Fix extended Schoenberg's work by considering a compactly supported function on $\mathbb{R}^n$ and its multi-integer translates. Let ϕ be a locally integrable function having compact support. For a sequence c on $\mathbb{Z}^n$, the semi-discrete convolution product $\phi *' c$ is the function given by

$$\phi *' c := \sum_{v \in \mathbb{Z}^n} \phi(\cdot - v)\, c(v).$$

If b and c both are sequences on $\mathbb{Z}^n$, we also denote by $b *' c$ the sum $\sum_{v \in \mathbb{Z}^n} b(\cdot - v)\, c(v)$. In [15] Strang and Fix introduced the concept of controlled approximation and successfully characterized the controlled L_2-approximation order by showing that ϕ provides controlled L_2-approximation of order k if and only if ϕ satisfies the conditions

(1°) $\hat{\phi}(0) \neq 0$ and

(2°) $D^\alpha \hat{\phi}(2\pi v) = 0$ for all $|\alpha| < k$ and $v \in \mathbb{Z}^n \setminus \{0\}$,

where we have used the multiindex notation. More precisely, we denote by $\mathbb{N}$ the set of nonnegative integers. An element $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}^n$ is called a multiindex. The length of α is defined to be $|\alpha| := \sum_{j=1}^n \alpha_j$, while the factorial of α is $\alpha! := \alpha_1! \cdots \alpha_n!$. For a multiindex α, $D^\alpha := D_1^{\alpha_1} \cdots D_n^{\alpha_n}$, where D_j is the partial derivative operator with respect to the jth coordinate, $j = 1, ..., n$. Now the conditions (1°) and (2°) together are called the Strang–Fix conditions of order k. We would like to emphasize that in the univariate case $(n = 1)$ these conditions already appeared in Schoenberg's work [13].

Strang and Fix [15] also considered multiinteger translates of a finite collection of compactly supported functions. To describe their results we need to introduce some notation. Let $\mathbb{R}^n$ be the n-dimensional real linear space with the norm $\|\cdot\|$ given by

$$\|x\| := \max_{1 \leqslant j \leqslant n} |x_j|, \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n.$$

If $\Omega \subseteq \mathbb{R}^n$ and $r \geqslant 0$, we denote by $B_r(\Omega)$ the *closed* ball of radius r around Ω, that is,

$$B_r(\Omega) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, \Omega) \leqslant r\}$$

with

$$\operatorname{dist}(x, \Omega) := \inf_{y \in \Omega} \|x - y\|.$$

When $\Omega = \{x\}$, we write $B_r(x)$ for $B_r(\Omega)$, and simply write B_r for $B_r(0)$. If f is a measurable function on a measurable subset Ω of $\mathbb{R}^n$, we denote by $\|f\|_p(\Omega)$ the quality $(\int_\Omega |f|^p\, dx)^{1/p}$. Similarly,

$$|f|_{k,p}(\Omega) := \sum_{|\alpha| = k} \|D^\alpha f\|_p(\Omega) \qquad \text{and} \qquad \|f\|_{k,p}(\Omega) := \sum_{|\alpha| \leqslant k} \|D^\alpha f\|_p(\Omega).$$

When Ω is omitted, the norm is understood to be taken over $\mathbb{R}^n$. Let $W_p^k = W_p^k(\mathbb{R}^n)$ be the usual Sobolev space equipped with the norm $\|\cdot\|_{k,p}$. A mapping from $\mathbb{Z}^n$ to $\mathbb{C}$ is called a sequence on $\mathbb{Z}^n$. The l_p-norm of a sequence $c = (c(v))_{v \in \mathbb{Z}^n}$ on Ω is defined to be

$$\|c\|_p(\Omega) := \left(\sum_{v \in \mathbb{Z}^n \cap \Omega} |c(v)|^p \right)^{1/p}.$$

Given $h > 0$, let σ_h be the scaling operator given by

$$\sigma_h f(x) := f(hx) \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

Let $\Phi = \{\phi_1, ..., \phi_N\}$ be a collection of compactly supported functions on $\mathbb{R}^n$. We say that Φ provides controlled L_p-approximation of order k if, for each $f \in W_p^k(\mathbb{R}^n)$, there exist sequences c_j^h ($j = 1, ..., N$; $h > 0$) such that the following inequalities hold for a constant C independent of h:

(i) $\|f - \sigma_{1/h}(\sum_{j=1}^N \phi_j *' c_j^h)\, h^{-n/p}\|_p \leqslant Ch^k |f|_{k,p}$;

(ii) $\|c_j^h\|_p \leqslant C\|f\|_p$, $j = 1, ..., N$.

The collection Φ is said to satisfy the Strang–Fix conditions of order k if there exist finitely supported sequences $b_j: \mathbb{Z}^n \to \mathbb{C}$ ($j = 1, ..., N$) such that the function $\sum_{j=1}^N \phi_j *' b_j$ satisfies the Strang–Fix conditions of order k. Strang and Fix claimed that Φ provides controlled L_2-approximation of order k if and only if Φ satisfies the Strang–Fix conditions of order k. Their claim had been in doubt for a long time and finally was disproved by Jia [10] using a counterexample.

In [9] Dahmen and Micchelli quoted the result of Strang and Fix in a modified form. They required (i) and (ii) to hold locally. We may say that Φ provides locally controlled L_p-approximation of order k if for each $f \in W_p^k(\mathbb{R}^n)$ there exist sequences c_j^h ($j = 1, ..., N$; $h > 0$) such that for any closed domain G,

(i′) $\|f - \sigma_{1/h}(\sum_{j=1}^N \phi_j *' c_j^h) h^{-n/p}\|_p(G) \leqslant Ch^k |f|_{k,p}(B_{rh}(G))$ and

(ii′) $\|c_j^h\|_p(h^{-1}G - \bigcup_{j=1}^N \operatorname{supp} \phi_j) \leqslant C\|f\|_p(B_{rh}(G))$, $j = 1, ..., N$,

hold for some constant C and r independent of h, G, and f. It turns out that Φ does satisfy the Strang–Fix conditions of order k if Φ provides locally controlled L_p-approximation of order k for any $p \in [1, \infty]$. This was proved by de Boor and Jia [3]. In fact, they proved an even stronger result by observing that it is the localness rather than the control that does the job. Thus they introduced the concept of local approximation order by saying that the collection Φ provides local L_p-approximation of order k, if for each $f \in W_p^k(\mathbb{R}^n)$, there exist sequences c_j^h ($j = 1, ..., N$) such that (i) and the following condition (iii) are satisfied:

(iii) There exist a constant r independent of h such that $\operatorname{dist}(vh, \operatorname{supp} f) > r \Rightarrow c_j^h(v) = 0$, $j = 1, ..., N$.

They proved that Φ provides local L_p-approximation of order k if and only if Φ satisfies the Strang–Fix conditions of order k.

Recently Light and Cheney [12] employed a finite collection $\Phi = \{\phi_1, ..., \phi_N\}$ of functions having global support to generate the approximations. Some assumptions were made about the rate of decay of these functions at ∞. Let $\lambda \in (0, 1)$ be fixed. For a positive integer k, let $E_k = E_k(\mathbb{R}^n)$ be the space of all functions f on $\mathbb{R}^n$ for which

$$\sup_{x \in \mathbb{R}^n} \{|f(x)| (1 + \|x\|)^{n+k+\lambda}\} < \infty.$$

Let Φ be a finite collection of functions in E_k. Light and Cheney [12] showed that Φ satisfies the Strang–Fix conditions of order k if and only if for each $f \in W_\infty^k(\mathbb{R}^n)$ there exist sequences b_j $(j = 1, ..., N)$ such that

$$\sup_{v \in \mathbb{Z}^n} \{|b_j(v)| (1 + \|v\|^{n+k+\lambda})\} < \infty, \qquad j = 1, ..., N,$$

and

$$\left\| f - \sigma_{1/h}\left(\sum_{j=1}^{N} \phi_j *' (b_j *' \sigma_h f) \right) \right\|_\infty \leqslant Ch^k |f|_{k,\infty}.$$

In this paper we improve and extend the results of Light and Cheney by introducing a modified notion of controlled approximation and characterizing the controlled approximation order. From now on we say that Φ provides *controlled L_p-approximation of order k* if, for each $f \in W_p^k(\mathbb{R}^n)$, there exist sequences c_j^h $(j = 1, ..., N;\ h > 0)$ such that (i), (ii), and (iii) hold. Thus our requirement for controlled approximation is stronger than that of Strang and Fix, but (seemingly) weaker than the requirement of Dahmen and Micchelli for locally controlled approximation.

In [12] Light and Cheney dealt with L_∞-approximation only. In this paper we deal not only with L_∞-approximation but also with L_p-approximation $(1 \leqslant p < \infty)$. It should be emphasized that the construction of L_p-approximation $(1 \leqslant p < \infty)$ requires much more work than that of L_∞-approximation. We also observe that the treatment for L_p-approximation in [3] was inadequate. The present paper gives us an opportunity to describe the L_p-approximation in details.

The main result of this paper is the following:

THEOREM 1.1. *Let Φ be a finite collection of elements in E_k. Then Φ provides controlled L_p-approximation $(1 \leqslant p \leqslant \infty)$ of order k if and only if Φ satisfies the Strang–Fix conditions of order k.*

It is not known whether the characterization of de Boor and Jia [3] for the local approximation order still holds in the present situation. We conjecture that this is the case, i.e., Φ provides local L_p-approximation of order k if and only if Φ satisfies the Strang–Fix conditions of order k.

The paper is organized as follows. In Section 2 we discuss the various equivalent forms of the Strang–Fix conditions. It is interesting to observe that the equivalence between those various forms can be proved without any consideration of approximation. In order to construct an L_p-approximation, an appropriate smoothing technique is needed. Section 3 is devoted to this smoothing technique and a related problem of local approximation. On the basis of the results in Section 3, we construct a global L_p-approximation in Section 4. Finally, in Section 5, we complete the proof of Theorem 1.1 by invoking a technique used in [3].

2. The Strang–Fix Conditions

The Strang–Fix conditions have various equivalent forms. In this section we show that the equivalence between these various forms can be proved without any consideration of approximation. The proof of the equivalence is based on Poisson's summation formula.

In what follows the Fourier transform of a function $\phi \in L^1(\mathbb{R}^n)$ is defined to be

$$\hat{\phi}(\xi) := \int_{\mathbb{R}^n} \phi(x) e^{-ix \cdot \xi}\, dx.$$

We denote by $\Pi = \Pi(\mathbb{R}^n)$ the linear space of all polynomials on $\mathbb{R}^n$, and by Π_k its subspace of all polynomials of (total) degree at most k. Following Light and Cheney [12], we denote by V_α the monomial given by $x \mapsto x^\alpha/\alpha!$, $x \in \mathbb{R}^n$. A mapping T on Π is called degree-reducing if for any $p \in \Pi$, Tp is a polynomial of degree less than the degree of p. If $p: x \mapsto \sum_\alpha a_\alpha x^\alpha$ is a polynomial, then $p(D)$ denotes the differential operator induced by p, i.e., $p(D) = \sum_\alpha a_\alpha D^\alpha$.

The following form of Poisson's summation formula may be found in Stein and Weiss [14, Chap. 7]. Suppose ϕ and its Fourier transform $\hat{\phi}$ are continuous on $\mathbb{R}^n$, and for some $\delta > 0$, there exists a constant $C > 0$ such that

$$|\phi(x)| \leq C(1 + \|x\|)^{-n-\delta}, \qquad x \in \mathbb{R}^n, \tag{2.1}$$

$$|\hat{\phi}(y)| \leq C(1 + \|y\|)^{-n-\delta}, \qquad y \in \mathbb{R}^n.$$

Then

$$\sum_{v \in \mathbb{Z}^n} \phi(v) = \sum_{v \in \mathbb{Z}^n} \hat{\phi}(2\pi v). \tag{2.2}$$

On the basis of this result, Dahmen and Micchelli in [8, Lemma 2.1] gave the following form of Poisson's summation formula: Suppose that $\phi \in C_c(\mathbb{R}^n)$ and

$$\hat{\phi}(2\pi\nu) = 0 \qquad \text{for} \quad \nu \in \mathbb{Z}^n \setminus \{0\}. \tag{2.3}$$

Then

$$\hat{\phi}(0) = \sum_{\nu \in \mathbb{Z}^n} \phi(\nu). \tag{2.4}$$

If ϕ is not compactly supported, then it is evident from the proof given in [8] that (2.4) still holds provided that ϕ satisfies the conditions (2.1) and (2.3).

In both [8] and [12], ϕ was assumed to be continuous. This would exclude the characteristic function of the unit cube, which is a typical spline function. Hence it is desirable to relax the continuity requirement. For this purpose Jia [11] introduced the concept of normal functions. A function ϕ on $\mathbb{R}^n$ is called normal if ϕ is locally integrable and for any $x \in \mathbb{R}^n$,

$$\phi(x) = \lim_{\varepsilon \to 0} \frac{1}{m(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \phi(y)\, dy, \tag{2.5}$$

where m denotes the Lebesgue measure. For a piecewise polynomial function ϕ, the limit on the right of (2.5) always exists, and this limit equals $\phi(x)$ if x is not on the mesh. The following form of Poisson's summation formula is what we will need.

THEOREM 2.1. *Suppose that ϕ is normal and satisfies* (2.1). *If the Fourier transform $\hat{\phi}$ satisfies* (2.3), *then* (2.4) *holds.*

Proof. For $\varepsilon > 0$, let

$$\phi_\varepsilon(x) := \frac{1}{m(B_\varepsilon(x))} \int_{B_\varepsilon(x)} \phi(y)\, dy.$$

Fix ε for the moment. Then $\phi_\varepsilon \in C(\mathbb{R}^n)$, and ϕ_ε satisfies (2.1). We observe that

$$\begin{aligned} \hat{\phi}_\varepsilon(2\pi\nu) &= \frac{1}{m(B_\varepsilon)} \int_{B_\varepsilon} \int_{\mathbb{R}^n} \phi(x+y) e^{-ix \cdot 2\pi\nu}\, dx\, dy \\ &= \hat{\phi}(2\pi\nu) \frac{1}{m(B_\varepsilon)} \int_{B_\varepsilon} e^{iy \cdot 2\pi\nu}\, dy. \end{aligned}$$

Hence

$$\begin{aligned} &\hat{\phi}_\varepsilon(2\pi\nu) = 0, \qquad \nu \in \mathbb{Z}^n \setminus \{0\}, \\ &\hat{\phi}_\varepsilon(0) = \hat{\phi}(0). \end{aligned}$$

Since ϕ_ε is continuous and satisfies (2.1) and (2.3), we have

$$\hat{\phi}_\varepsilon(0) = \sum_{v \in \mathbb{Z}^n} \phi_\varepsilon(v). \tag{2.6}$$

By the normality of ϕ,

$$\phi(v) = \lim_{\varepsilon \to 0} \phi_\varepsilon(v), \qquad v \in \mathbb{Z}^n.$$

Letting $\varepsilon \to 0$ in (2.6), we obtain the desired result. ∎

Let now ϕ be a normal function in $E_k(\mathbb{R}^n)$. We consider the action of the operator $\phi *'$ on Π_{k-1}. Note that for any $p \in \Pi_{k-1}$, $\phi *' p = \sum_{v \in \mathbb{Z}^n} \phi(\cdot - v)\, p(v)$ is well defined. The following theorem and its corollaries can be proved on the basis of Theorem 2.1 (the Poisson summation formula).

THEOREM 2.2. *Let ϕ be a normal function in $E_k(\mathbb{R}^n)$. The mapping $\phi *'$ maps Π_{k-1} into itself if and only if the Fourier transform $\hat{\phi}$ satisfies*

$$p(-iD)\,\hat{\phi}(2\pi v) = 0 \qquad \textit{for all} \quad p \in \Pi_{k-1} \quad \textit{and} \quad v \in \mathbb{Z}^n \setminus \{0\}. \tag{2.7}$$

COROLLARY 2.3. *The mapping $\phi *'$ is an isomorphism on Π_{k-1} if and only if ϕ satisfies the Strang–Fix conditions of order k, that is, $\hat{\phi}$ satisfies (2.7) and $\hat{\phi}(0) \neq 0$. If, in addition to (2.7), $\hat{\phi}$ satisfies $\hat{\phi}(0) = 1$, then $1 - \phi *'$ is a degree-reducing mapping on Π_{k-1}.*

COROLLARY 2.4. *The mapping $\phi *'$ is the identity on Π_{k-1} if and only if in addition to (2.7) $\hat{\phi}$ satisfies*

$$p(-iD)\,\hat{\phi}(0) = p(0) \qquad \textit{for all} \quad p \in \Pi_{k-1}.$$

Remark. When ϕ is compactly supported, Theorem 2.2 and its corollaries were first proved by Strang and Fix in [15]. See [1, 2, 8, 11] for various extensions of Strang and Fix's results. The proofs given in [1, 2, 8, 11] can be carried over verbatim in the present situation.

We consider now a finite collection $\Phi = \{\phi_1, ..., \phi_N\}$ of normal functions in E_k. Recall that Φ is said to satisfy the Strang–Fix conditions of order k if there exist finitely supported sequences b_j $(j = 1, ..., N)$ such that $\phi = \sum_{j=1}^N \phi_j *' b_j$ satisfies the Strang–Fix conditions of order k. When $\phi_1, ..., \phi_N$ are compactly supported, Strang and Fix [15] gave several equivalent forms of their conditions. Their results can be extended to the case in which $\phi_1, ..., \phi_N$ are in E_k.

THEOREM 2.5. *Let $\Phi = \{\phi_1, ..., \phi_N\}$ be a collection of normal functions in E_k. Then the following conditions are equivalent:*

(I) *There are functions ψ_α ($|\alpha| < k$) in* span(Φ) *such that $\hat{\psi}_0(0) = 1$ and*

$$\sum_{\beta \leq \alpha} V_\beta(-iD)\, \hat{\psi}_{\alpha-\beta}(2\pi\nu) = 0 \qquad \text{for} \quad \nu \in \mathbb{Z}^n \backslash \{0\} \quad \text{and} \quad |\alpha| < k. \tag{2.8}$$

(II) *There exist ψ_α ($|\alpha| < k$) in* span(Φ) *such that*

$$V_\alpha - \sum_{\beta \leq \alpha} \psi_{\alpha-\beta} *' V_\beta \in \Pi_{|\alpha|-1}.$$

(III) *There exist finitely supported sequences b_j such that $\phi = \sum_{j=1}^N \phi_j *' b_j$ satisfies the condition* (2.7) *and $\hat{\phi}(0) \neq 0$.*

In fact, even in the case where $\phi_1, ..., \phi_N$ are compactly supported, the proof given by Strang and Fix in [15] was incomplete. The first complete proof of this result was given by de Boor and Jia in [3]. Their result was extended by Light and Cheney [12] to the present theorem.

Here we point out that the equivalence of the properties (I), (II), and (III) can be proved without any consideration of approximation. For this purpose we only need to supplement a proof for (III) $\Rightarrow$ (I). Suppose $\phi = \sum_{j=1}^N \phi_j *' b_j$ satisfies $\hat{\phi}(0) = 1$ and condition (2.7). Then we have

$$\hat{\phi}(\xi) = \sum_{j=1}^N \sum_{\gamma \in \mathbb{Z}^n} b_j(\gamma) e^{-i\gamma \cdot \xi} \hat{\phi}_j(\xi), \qquad \xi \in \mathbb{R}^n. \tag{2.9}$$

Let

$$\psi_\alpha := \sum_{j=1}^N \sum_{\gamma \in \mathbb{Z}^n} b_j(\gamma)\, V_\alpha(-\gamma) \phi_j, \qquad |\alpha| < k.$$

We claim that $(\psi_\alpha)_{|\alpha|<k}$ satisfies $\hat{\psi}_0(0) = 1$ and the condition (2.8). Indeed, it follows from (2.9) that

$$1 = \hat{\phi}(0) = \sum_{j=1}^N \sum_{\gamma \in \mathbb{Z}^n} b_j(\gamma)\, \hat{\phi}_j(0).$$

Hence $\hat{\psi}_0(0) = 1$, as desired. Moreover, for $|\alpha| < k$ and $\nu \in \mathbb{Z}^n \backslash \{0\}$, an application of the Leibniz differentiation formula gives

$$\begin{aligned} 0 &= V_\alpha(-iD)\, \hat{\phi}(2\pi\nu) \\ &= \sum_{\beta \leq \alpha} \left\{ \sum_{j=1}^N \sum_{\gamma \in \mathbb{Z}^n} V_{\alpha-\beta}(-\gamma)\, b_j(\gamma) \right\} V_\beta(-iD)\, \hat{\phi}_j(2\pi\nu) \\ &= \sum_{\beta \leq \alpha} V_\beta(-iD)\, \hat{\psi}_{\alpha-\beta}(2\pi\nu). \end{aligned}$$

This completes the proof of Theorem 2.5.

If $\Phi = \{\phi_1, ..., \phi_N\}$ satisfies the Strang–Fix conditions of order k, then there exist finitely supported sequences b_j $(j = 1, ..., N)$ such that the function $\phi := \sum_{j=1}^{N} \phi_j *' b_j$ induces a degree-reducing mapping $1 - \phi *'$ on Π_{k-1}. Light and Cheney [12] proved that there exists a sequence b such that the function $\psi := \phi *' b$ induces an identity $\psi *'$ on Π_{k-1}. Using this ψ, they constructed an L_∞-approximation as follows. For $f \in W_\infty^k(\mathbb{R}^n)$, let

$$s_h(f) := \sigma_{1/h}(\psi *' \sigma_h f). \tag{2.10}$$

They showed that there exists a constant C such that

$$\|f - s_h(f)\|_\infty \leqslant Ch^k |f|_{k,\infty} \qquad \text{for all} \quad f \in W_\infty^k(\mathbb{R}^n) \text{ and } h > 0. \tag{2.11}$$

The sequence b they used, however, is not finitely supported, hence their method does not produce a local approximation in the sense of de Boor and Jia [3]. To overcome this difficulty we prove the following lemma showing that b can be so chosen that it is supported on the set $\{\alpha \in \mathbb{N}^n: |\alpha| \leqslant k-1\}$ (see [1, 4]). In the following, for $x \in \mathbb{R}^n$, $[x]$ denotes the functional of point-evaluation at x, and T_x denotes the translation operator given by $T_x(f) = f(\cdot + x)$.

LEMMA 2.6. *Let $\phi \in E_k$ be a normal function such that $1 - \phi *'$ is a degree-reducing mapping on Π_{k-1}. Then there exists a sequence b supported on $\{\alpha \in \mathbb{N}^n: |\alpha| \leqslant k-1\}$ such that the mapping $\psi *'$ induced by $\psi = \phi *' b$ is the identity on Π_{k-1}.*

Proof. The operator $L := \phi *'|_{\Pi_{k-1}}$ is an isomorphism on Π_{k-1}. Moreover L commutes with every translation operator T_x, $x \in \mathbb{R}^n$. Let L^{-1} be its inverse operator. Then L^{-1} also commutes with translation operators, and $[0]L^{-1}$ is a linear functional on Π_{k-1}. Let Λ be the span of linear functionals $[\alpha]$, $\alpha \in \mathbb{N}^n$, $|\alpha| \leqslant k-1$. Then Λ and Π_{k-1} are dual to each other with respect to the bilinear function given by

$$\langle [\alpha], p \rangle := p(\alpha)$$

(see, e.g., [5]). Hence there exist complex numbers a_α $(\alpha \in \mathbb{N}^n$ and $|\alpha| \leqslant k-1)$ such that

$$[0]L^{-1} = \sum_{|\alpha| \leqslant k-1} a_\alpha [\alpha].$$

For any $p \in \Pi_{k-1}$ and $x \in \mathbb{R}^n$, $p(\cdot + x)$ is also in Π_{k-1}. Hence the above equation yields

$$[0]L^{-1}(p(\cdot + x)) = \sum_{|\alpha| \leqslant k-1} a_\alpha [\alpha](p(\cdot + x)).$$

Since L^{-1} commutes with T_x, we have

$$(L^{-1}p)(x) = \sum_{|\alpha| \leqslant k-1} a_\alpha T_\alpha p(x) \qquad \text{for all} \quad x \in \mathbb{R}^n.$$

It follows that

$$L^{-1} = \sum_{|\alpha| \leqslant k-1} a_\alpha T_\alpha.$$

Let $b: \mathbb{Z}^n \to \mathbb{C}$ be given by

$$b(\beta) = \begin{cases} a_\beta, & \text{if } \beta \in \mathbb{N}^n \text{ and } |\beta| \leqslant k-1; \\ 0, & \text{otherwise.} \end{cases}$$

Then for any $p \in \Pi_{k-1}$

$$(\phi *' b) *' p = \left(\sum_{|\alpha| \leqslant k-1} a_\alpha T_\alpha \phi \right) *' p = L^{-1}(\phi *' p) = L^{-1}(Lp) = p. \quad \blacksquare$$

Now with the help of Lemma 2.6 the result of Light and Cheney in [12] says that if Φ satisfies the Strang–Fix conditions of order k, then Φ provides controlled L_∞-approximation of order k. Note that the point-evaluation functionals $[hv]$ are used in (2.10). In general, point-evaluation functionals are not continuous on $W_p^k(\mathbb{R}^n)$ $(1 \leqslant p < \infty)$. Hence a suitable smoothing technique must be worked out in order to construct an L_p-approximation. This is the content of the next section.

3. Local Approximation

Let ψ be an element of $C_c^\infty(\mathbb{R}^n)$ such that $\operatorname{supp} \psi \subset B_1(0)$, $\psi \geqslant 0$, and $\int \psi = 1$. Set

$$\psi_h := \psi(\cdot/h)/h^n, \qquad h > 0.$$

For a given function $f \in L_p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ and $h > 0$, consider the function

$$f_h(x) := \int_{\mathbb{R}^n} (f - \nabla_u^k f)(x)\, \psi_h(u)\, du, \qquad x \in \mathbb{R}^n, \tag{3.1}$$

where ∇_u denotes the difference operator given by

$$\nabla_u f := f - f(\cdot - u).$$

Correspondingly, we denote by D_u the directional derivative operator in direction u, i.e.,

$$D_u = \sum_{j=1}^{n} u_j D_j \qquad \text{for} \quad u = (u_1, \ldots, u_n) \in \mathbb{R}^n.$$

In the univariate case ($n = 1$), such a smoothing technique was employed by DeVore [7] in studying degree of approximation.

THEOREM 3.1. *For $f \in L_p(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$, the functions f_h are C^∞-smooth. Moreover, there exists a constant C depending only on k and n such that for any measurable set $\Omega \subseteq \mathbb{R}^n$*

(a) $\|f_h\|_p(\Omega) \leqslant C \|f\|_p(B_{kh}(\Omega))$;

(b) $\|f_h\|_\infty(\Omega) \leqslant C h^{-n/p} \|f\|_p(B_{kh}(\Omega))$ $(1 \leqslant p < \infty)$;

(c) $(\sum_{v \in \mathbb{Z}^n} |f_h(hv)|^p)^{1/p} \leqslant C h^{-n/p} \|f\|_p$ $(1 \leqslant p < \infty)$.

Proof. From the well-known expression

$$\nabla_u^k f = \sum_{m=0}^{k} (-1)^m \binom{k}{m} f(\cdot - mu),$$

we deduce that

$$f - \nabla_u^k f = \sum_{m=1}^{k} (-1)^{m-1} \binom{k}{m} f(\cdot - mu). \tag{3.2}$$

Observe that for $m \geqslant 1$

$$\int_{\mathbb{R}^n} f(\cdot - mu)\, \psi_h(u)\, du = \int_{\mathbb{R}^n} f(\cdot - u)\, \psi_h(u/m)/m^n\, du = f * \psi_h(\cdot/m)/m^n. \tag{3.3}$$

From (3.1)–(3.3) we see that the functions f_h are C^∞-smooth, because ψ_h are C^∞ functions for any $h > 0$.

Part (a) of this theorem can be proved by applying the generalized Minkowski inequality to the integral in (3.3). For $1 \leqslant m \leqslant k$, we have

$$\left\| \int_{\mathbb{R}^n} f(\cdot - mu)\, \psi_h(u)\, du \right\|_p (\Omega) \leqslant \int_{\mathbb{R}^n} \|f(\cdot - mu)\|_p\, (\Omega)\, \psi_h(u)\, du$$

$$\leqslant \|f\|_p\, (B_{kh}(\Omega)) \int_{\mathbb{R}^n} \psi_h(u)\, du = \|f\|_p\, (B_{kh}(\Omega)).$$

It follows that

$$\|f_h\|_p(\Omega) \leqslant C \|f\|_p (B_{kh}(\Omega))$$

with

$$C \leqslant \sum_{m=1}^{k} \binom{k}{m} = 2^k - 1.$$

The proof of part (b) relies on Hölder's inequality. Let q be the exponential conjugate to p, i.e., $1/q + 1/p = 1$. Note that $\operatorname{supp} \psi_h \subseteq B_h$. By Hölder's inequality, we have

$$\left| \int_{\mathbb{R}^n} f(x-mu)\, \psi_h(u)\, du \right| \leqslant \left(\int_{B_h} |f(x-mu)|^p\, du \right)^{1/p} \left(\int_{B_h} (\psi_h(u))^q\, du \right)^{1/q}. \tag{3.4}$$

For the first integral on the right of (3.4) we have the following estimate:

$$\left(\int_{B_h} |f(x-mu)|^p\, du \right)^{1/p} \leqslant \|f\|_p(B_{kh}(x)), \qquad \text{for} \quad 0 \leqslant m \leqslant k.$$

Making change of variables in the second integral gives

$$\left(\int_{B_h} (\psi_h(u))^q\, du \right)^{1/q} = \left(h^n \int_{B_1} (\psi(u))^q\, du \right)^{1/q} \Big/ h^n \leqslant C h^{-n/p},$$

where $C = \sup_{u \in \mathbb{R}^n} \{\psi(u)\}$. This proves part (b).

In particular, part (b) implies that

$$\sum_{v \in \mathbb{Z}^n} |f_h(hv)|^p \leqslant C^p h^{-n} \sum_{v \in \mathbb{Z}^n} \int_{B_{kh}(vh)} |f(x)|^p\, dx.$$

We observe that every $x \in \mathbb{R}^n$ is covered by at most $(2k+1)^n$ balls $B_{kh}(vh)$ $(v \in \mathbb{Z}^n)$. Thus the proof of part (c) will be complete once we prove the following lemma.

LEMMA 3.2. *Let $\mathscr{E}$ be a collection of measurable subsets of $\mathbb{R}^n$ and M a positive integer. Suppose that every point $x \in \mathbb{R}^n$ is covered by at most M sets from $\mathscr{E}$. Then for any $g \in L_1(\mathbb{R}^n)$,*

$$\sum_{E \in \mathscr{E}} \int_E |g(x)|\, dx \leqslant M \int_{\mathbb{R}^n} |g(x)|\, dx.$$

Proof. For each $E \in \mathscr{E}$, let χ_E denote its characteristic function. Since every point $x \in \mathbb{R}^n$ is covered by at most M sets from $\mathscr{E}$, we have $\sum_{E \in \mathscr{E}} \chi_E(x) \leqslant M$ for all $x \in \mathbb{R}^n$. Therefore,

$$\sum_{E \in \mathscr{E}} \int_E |g(x)|\, dx \leqslant \int_{\mathbb{R}^n} |g(x)| \sum_{E \in \mathscr{E}} \chi_E(x)\, dx \leqslant M \int_{\mathbb{R}^n} |g(x)|\, dx,$$

as desired. ∎

Remark. Sometimes it is desirable to relax the smoothness requirement for the kernel ψ. If $\psi \in C^k$, then f_h are also C^k-smooth, and the properties (a), (b), and (c) of Theorem 3.1 still hold.

The following theorem tells us that f_h is a good approximation to f when h is small.

THEOREM 3.3. *There exists a constant C depending only on k and n such that for any* $f \in W_p^k(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ *and any measurable set* $\Omega \subseteq \mathbb{R}^n$,

$$|f - f_h|_{l,p}(\Omega) \leqslant Ch^{k-l}|f|_{k,p}(B_{kh}(\Omega)), \qquad 0 \leqslant l \leqslant k. \tag{3.5}$$

Proof. If $l = k$, then (3.5) is a consequence of Theorem 3.1. Thus we only need to deal with the case $0 \leqslant l < k$. First we assume that f is C^k-smooth. It suffices to estimate $\|D^\beta(f - f_h)\|_p$ for each $\beta \in \mathbb{N}^n$, $|\beta| = l < k$. Since

$$(f - f_h)(x) = \int_{\mathbb{R}^n} \nabla_u^k f(x)\, \psi_h(u)\, du,$$

we have

$$D^\beta(f - f_h)(x) = \int_{\mathbb{R}^n} \nabla_u^k D^\beta f(x)\, \psi_h(u)\, du.$$

Given a positive integer r, let M_r be the B-spline given by the rule

$$M_r(t) := \nabla^r(\cdot - t)_+^{r-1}/(r-1)!, \tag{3.6}$$

where ∇ is the difference operator defined by $\nabla g := g - g(\cdot - 1)$. Then M_r is supported on the interval $[0, r]$. Moreover, by Peano's theorem (see, e.g., [6, p. 70]),

$$\nabla^r g = \int_0^r (D^r g)(\cdot - t)\, M_r(t)\, dt, \qquad \text{for all} \quad g \in C^r(\mathbb{R}).$$

Hence

$$\nabla_u^k D^\beta f = \nabla_u^{k-l} \nabla_u^l D^\beta f = \int_0^{k-l} D_u^{k-l} \nabla_u^l D^\beta f(\cdot - tu)\, M_{k-l}(t)\, dt.$$

It follows that

$$D^\beta(f - f_h)(x) = \int_{\mathbb{R}^n} \int_0^{k-l} \nabla_u^l D_u^{k-l} D^\beta f(\cdot - tu)\, M_{k-l}(t)\, \psi_h(u)\, dt\, du.$$

Applying the generalized Minkowski inequality to the above integral, we obtain

$$\|D^\beta(f-f_h)\|_p(\Omega) \leqslant \int_{\mathbb{R}^n}\int_0^{k-l} \|\nabla_u^l D_u^{k-l} D^\beta f(\cdot - tu)\|_p(\Omega)\, M_{k-l}(t)\, \psi_h(u)\, dt\, du. \tag{3.7}$$

Note that $D_u=\sum_{j=1}^n u_j D_j$ for $u=(u_1, ..., u_n)$. Hence for $u\in \operatorname{supp}\psi_h \subseteq B_h$ and $0\leqslant t\leqslant k-l$, we have

$$\|\nabla_u^l D_u^{k-l} D^\beta f(\cdot - tu)\|_p(\Omega) \leqslant Ch^{k-l}|f|_{k,p}(B_{kh}(\Omega)), \tag{3.8}$$

where C is a constant depending only on k and n. Since both the integrals $\int_0^{k-l} M_{k-l}(t)\,dt$ and $\int_{\mathbb{R}^n}\psi_h(u)\,du$ are equal to 1, hence (3.7) and (3.8) together yield the following estimate:

$$\|D^\beta(f-f_h)\|_p(\Omega) \leqslant Ch^{k-l}|f|_{k,p}(B_{kh}(\Omega)). \tag{3.9}$$

When $1\leqslant p<\infty$, $C^k(\mathbb{R}^n)\cap W_p^k(\mathbb{R}^n)$ is dense in $W_p^k(\mathbb{R}^n)$, so the estimate (3.9) holds for any $f\in W_p^k(\mathbb{R}^n)$. In the case $p=\infty$, we may assume without loss that the measure of Ω is finite and positive. Then any $g\in L_\infty(\Omega)$ is also in $L_p(\Omega)$ $(1\leqslant p<\infty)$. Moreover, $\|g\|_p(\Omega)$ converges to $\|g\|_\infty$ as $p\to\infty$. Thus, letting $p\to\infty$ in (3.9), we conclude that (3.9) is also true for $p=\infty$ and $f\in W_\infty^k(\mathbb{R}^n)$. ∎

4. Global Approximation

Let ϕ be a normal function in $E_k(\mathbb{R}^n)$. Then there is $\lambda>0$ such that

$$|\phi(x)|\leqslant C(1+\|x\|)^{-n-k-\lambda} \qquad \text{for all} \quad x\in\mathbb{R}^n,$$

where C is a constant depending only on ϕ. In this section we construct an L_p-approximation $(1\leqslant p\leqslant\infty)$ using translates of ϕ. In what follows C means a constant depending only on k, n, λ, and ϕ. In particular, C is independent of p. But C may be different in different contexts. Our construction is motivated by the work of Dahmen and Micchelli [9].

Theorem 4.1. *Let $\phi\in E_k(\mathbb{R}^n)$ be a normal function. For $f\in W_p^k(\mathbb{R}^n)$ $(1\leqslant p\leqslant\infty)$ and $h>0$, set*

$$s_h(x) := \sum_{v\in\mathbb{Z}^n} f_h(hv)\,\phi(x/h-v), \qquad x\in\mathbb{R}^n,$$

where f_h are given by (3.1). *If $\phi *'$ is the identity on Π_{k-1}, then*

$$\|f - s_h\|_p \leqslant C|f|_{k,p} h^k.$$

Proof. In the case $p = \infty$, the above estimate was already proved by Light and Cheney [12] for $s_h := \sigma_{1/h}(\phi *' \sigma_h f)$ (see (2.10) and (2.11)). Their proof is also valid in the present situation. In the following we assume that $1 \leqslant p < \infty$. By Theorem 3.3, we have

$$\|f - f_h\|_p \leqslant C|f|_{k,p} h^k.$$

Hence it remains to show that

$$\|f_h - s_h\|_p \leqslant C|f|_{k,p} h^k.$$

Let

$$G_{\alpha,h} := (\alpha + [0, 1]^n)h, \qquad \alpha \in \mathbb{Z}^n.$$

We estimate $|f_h(x) - s_h(x)|$ for $x \in G_{\alpha,h}$. Fix α and $x \in G_{\alpha,h}$ for the moment. Let u be the Taylor polynomial of f_h of degree $k-1$ about x. In particular, $f_h(x) = u(x)$. Since $\phi *'$ is the identity on Π_{k-1}, we have

$$u = \sum_{v \in \mathbb{Z}^n} u(hv)\, \phi(\cdot/h - v).$$

It follows that

$$\begin{aligned} f_h(x) - s_h(x) &= u(x) - s_h(x) \\ &= \sum_{v \in \mathbb{Z}^n} (u(hv) - f_h(hv))\, \phi(x/h - v) \\ &= \sum_{v \in \mathbb{Z}^n} (u(h\alpha + hv) - f_h(h\alpha + hv))\, \phi(x/h - \alpha - v). \qquad (4.1) \end{aligned}$$

Since $\phi \in E_k$ and $x/h - \alpha \in [0, 1]^n$ for $x \in G_{\alpha,h}$, we have

$$|\phi(x/h - \alpha - v)| \leqslant C(1 + \|v\|)^{-k-n-\lambda} \qquad \text{for all} \quad x \in G_{\alpha,h}. \qquad (4.2)$$

We use Taylor's formula to estimate $|f_h(h\alpha + hv) - u(h\alpha + hv)|$. For simplicity write $y := h\alpha + hv$ and $\xi_{v,t} := (1-t)x + ty$. Then by Taylor's theorem we have

$$u(hv + h\alpha) - f_h(hv + h\alpha) = -\int_0^1 D_{y-x}^k f_h(\xi_{v,t})(1-t)^{k-1}/(k-1)!\, dt.$$

Note that

$$\|y - x\| = \|hv + h\alpha - x\| \leqslant h(1 + \|v\|) \qquad \text{for} \quad x \in G_{\alpha,h}.$$

Hence

$$|D^k_{y-x} f_h(\xi_{\nu,t})| \leqslant Ch^k(1+\|\nu\|)^k F_h(\xi_{\nu,t}),$$

where

$$F_h := \sum_{|\beta|=k} |D^\beta f_h|. \tag{4.3}$$

From the above estimates we see that

$$|u(h\nu+h\alpha)-f_h(h\nu+h\alpha)| \leqslant Ch^k(1+\|\nu\|)^k \int_0^1 F_h(\xi_{\nu,t})\,dt. \tag{4.4}$$

Thus (4.1), (4.2), and (4.4) together yield the following estimate:

$$|f_h(x)-s_h(x)| \leqslant Ch^k \sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda} \int_0^1 F_h(\xi_{\nu,t})\,dt.$$

Let q be the exponential conjugate to p, i.e., $1/q+1/p=1$. By Hölder's inequality, we have

$$\sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda} \int_0^1 F_h(\xi_{\nu,t})\,dt$$

$$\leqslant \left\{\sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda}\right\}^{1/q} \left\{\sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda} \left(\int_0^1 F_h(\xi_{\nu,t})\,dt\right)^p\right\}^{1/p}$$

The first factor on the right of the above inequality is bounded from above by a constant independent of q. To estimate the second factor, we use Hölder's inequality again to obtain

$$\int_0^1 F_h(\xi_{\nu,t})\,dt \leqslant \left(\int_0^1 1^q\,dt\right)^{1/q} \left(\int_0^1 F_h^p(\xi_{\nu,t})\,dt\right)^{1/p} = \left(\int_0^1 F_h^p(\xi_{\nu,t})\,dt\right)^{1/p}.$$

Consequently, we have

$$|f_h(x)-s_h(x)| \leqslant Ch^k \left\{\sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda} \int_0^1 F_h^p(\xi_{\nu,t})\,dt\right\}^{1/p},$$

$$\text{for all} \quad x\in G_{\alpha,h}.$$

It follows that

$$\|f_h-s_h\|_p^p = \sum_{\alpha\in\mathbb{Z}^n} \int_{G_{\alpha,h}} |f_h(x)-s_h(x)|^p\,dx$$

$$\leqslant C^p h^{kp} \sum_{\nu\in\mathbb{Z}^n} (1+\|\nu\|)^{-n-\lambda} \sum_{\alpha\in\mathbb{Z}^n} \int_{G_{\alpha,h}} \int_0^1 F_h^p(\xi_{\nu,t})\,dt\,dx. \tag{4.5}$$

To estimate the above sum over $\alpha \in \mathbb{Z}^n$, we divide the unit interval $[0, 1]$ into $1 + \|v\|$ equal parts by setting

$$I_j := \{t \in \mathbb{R} : j/(1 + \|v\|) \leqslant t \leqslant (j+1)/(1 + \|v\|)\}, \qquad j = 0, ..., \|v\|.$$

Thus

$$\int_0^1 F_h^p(\xi_{v,t})\, dt = \sum_{j=0}^{\|v\|} \int_{I_j} F_h^p(\xi_{v,t})\, dt.$$

Since $x \in G_{\alpha,h}$, for $t \in I_j$, we have

$$\xi_{v,j} = (1-t)x + t(h\alpha + hv) = ((1-t)x + t(h\alpha)) + thv \in G_{\alpha,h} + hvI_j,$$

while the length of each I_j is $1/(1 + \|v\|)$. Hence

$$\int_{I_j} F_h^p(\xi_{v,t})\, dt \leqslant 1/(1 + \|v\|)\, \|F_h\|_\infty^p (G_{\alpha,h} + hvI_j).$$

To estimate $\|F_h\|_\infty$, we set $F := \sum_{|\beta| = k} |D^\beta f|$ in correspondence to (4.3). Applying Theorem 3.1 (b) to each $D^\beta f$ ($|\beta| = k$) and $\Omega := G_{\alpha,h} + hvI_j$, and then adding them up, we get

$$\|F_h\|_\infty^p (G_{\alpha,h} + hvI_j) \leqslant C^p h^{-n} \|F\|_p^p (G_{\alpha,h} + hvI_j + B_{kh}).$$

Putting these estimates together, we obtain

$$\sum_{\alpha \in \mathbb{Z}^n} \int_{G_{\alpha,h}} \int_0^1 F_h^p(\xi_{v,t})\, dt\, dx \leqslant \frac{C^p}{1 + \|v\|} \sum_{j=0}^{\|v\|} \sum_{\alpha \in \mathbb{Z}^n} \|F\|_p^p (G_{\alpha,h} + hvI_j + B_{kh}), \tag{4.6}$$

where we have used the fact that the Lebesgue measure of $G_{\alpha,h}$ is h^n. We observe that

$$G_{\alpha,h} + hvI_j + B_{kh} \subseteq h\alpha + hvj/(1 + \|v\|) + B_{(k+2)h};$$

hence every point $x \in \mathbb{R}^n$ is covered by at most $(2k+5)^n$ sets from the collection $\{G_{\alpha h} + hvI_j + B_{(k+2)h} : \alpha \in \mathbb{Z}^n\}$. Therefore, by Lemma 3.2, we obtain

$$\sum_{\alpha \in \mathbb{Z}^n} \|F\|_p^p (G_{\alpha,h} + hvI_j + B_{kh}) \leqslant (2k+5)^n \|F\|_p^p,$$

where

$$\|F\|_p^p = \int_{\mathbb{R}^n} F^p(x)\, dx = \int_{\mathbb{R}^n} \sum_{|\beta| = k} |D^\beta f(x)|^p\, dx \leqslant |f|_{k,p}^p.$$

This together with (4.5) and (4.6) gives the desired estimate:

$$\|f_h - s_h\|_p \leqslant Ch^k |f|_{k,p}. \quad \blacksquare$$

Let us summarize what we have proved. Let $\Phi = \{\phi_1, ..., \phi_N\}$ be a finite collection of normal functions in $E_k(\mathbb{R}^n)$. In Section 2 we showed that if Φ satisfies the Strang–Fix conditions of order k, then there exist finitely supported sequences b_j $(j = 1, ..., N)$ such that the mapping $\phi *'$ induced by $\phi := \sum_{j=1}^N \phi_j *' b_j$ is the identity on Π_{k-1}. In this section we have shown that for $f \in W_p^k(\mathbb{R}^n)$ $(1 \leqslant p \leqslant \infty)$ and $h > 0$, the function

$$s_h := \sigma_{1/h}(\phi *' \sigma_h f_h)$$

satisfies

$$\|f - s_h\|_p \leqslant Ch^k |f|_{k,p}.$$

In the above expression of s_h, substituting ϕ by $\sum_{j=1}^N \phi_j *' b_j$, we obtain

$$s_h = \sum_{j=1}^N \sigma_{1/h}(\phi_j *' c_j^h) h^{-n/p},$$

where

$$c_j^h = (b_j *' \sigma_h f_h) h^{n/p}, \qquad j = 1, ..., N.$$

Since b_j $(j = 1, ..., N)$ are finitely supported, the sequences c_j^h satisfy the condition (iii) in Section 1. Moreover, by Theorem 3.1 (c), c_j^h also satisfy the condition (ii) there. Thus the sufficiency part of Theorem 1.1 has been proved. The necessity part of Theorem 1.1 will be proved in the next section.

5. Characterization of the Controlled Approximation Order

We restate the necessity part of Theorem 1.1 as follows:

Theorem 5.1. *Let $\Phi = \{\phi_1, ..., \phi_N\}$ be a collection of normal functions in $E_k(\mathbb{R}^n)$. Let p be a fixed real number, $1 \leqslant p \leqslant \infty$. If for each $f \in W_p^k(\mathbb{R}^n)$, there exist sequences c_j^h $(j = 1, ..., N)$ such that*

(i) $\|f - \sigma_{1/h}(\sum_{j=1}^N \phi_j *' c_j^h) h^{-n/p}\|_p \leqslant Ch^k |f|_{k,p}$,

(ii) $\|c_j^h\|_p \leqslant C \|f\|_p$, $j = 1, ..., N$, *and*

(iii) $\operatorname{dist}(hv, \operatorname{supp} f) > r \Rightarrow c_j^h(v) = 0$, $j = 1, ..., N$, *where C and r are positive constants independent of h, then Φ satisfies the Strang–Fix conditions of order k.*

Proof. The proof goes along the line of [3]. We approximate a tensor product of univariate B-splines—namely, the function

$$u(x) := \prod_{j=1}^{n} M_{k+1}\left(x_j - \frac{k+1}{2}\right), \qquad x = (x_1, ..., x_n) \in \mathbb{R}^n, \tag{5.1}$$

where M_r is the B-spline given by (3.6) (see [13]). Since $u \in W_p^k(\mathbb{R}^n)$, we can find sequences c_j^h $(j = 1, ..., N; h > 0)$ so that the conditions (i), (ii), and (iii) are satisfied. Let

$$u_h := h^{-n/p} \sigma_{1/h} \left(\sum_{j=1}^{N} \phi_j *' c_j^h \right)$$

and set $g := u - u_h$. Then the property (i) implies

$$\|g\|_p \leqslant Ch^k.$$

We claim that

$$\|D^\alpha \hat{g}\|_\infty \leqslant Ch^k \qquad \text{for} \quad |\alpha| = m \leqslant k \quad \text{and} \quad 0 < h \leqslant 1. \tag{5.2}$$

To this end we apply the differential operator D^α to $\hat{g}$ and obtain

$$D^\alpha \hat{g}(\xi) = \int_{\mathbb{R}^n} (-ix)^\alpha \, g(x) e^{-i\xi \cdot x} \, dx.$$

It follows that

$$\|D^\alpha \hat{g}\|_\infty \leqslant \int_{\mathbb{R}^n} \|x\|^m \, |g(x)| \, dx =: J \qquad \text{for} \quad |\alpha| = m.$$

Let $R := r + k$. Since the support of u is included in the ball B_k, by the property (iii) we have

$$c_j^h(v) = 0 \qquad \text{for} \quad \|vh\| > R. \tag{5.3}$$

Write

$$J = \int_{\|x\| < 2R} \|x\|^m \, |g(x)| \, dx + \int_{\|x\| \geqslant 2R} \|x\|^m \, |g(x)| \, dx =: J_1 + J_2.$$

Applying Hölder's inequality to the first integral J_1, we obtain

$$J_1 \leqslant \left(\int_{\|x\| < 2R} (\|x\|^m)^q \, dx \right)^{1/q} \left(\int_{\|x\| < 2R} |g(x)|^p \, dx \right)^{1/p} \leqslant C \|g\|_p \leqslant Ch^k.$$

For the second integral J_2 we observe that

$$\|x\| \geqslant 2R \quad \text{and} \quad \|vh\| \leqslant R \Rightarrow u(x) = 0 \quad \text{and} \quad \|x/h - v\| \geqslant \|x\|/(2h). \tag{5.4}$$

It follows from (5.3) and (5.4) that for $\|x\| \geqslant 2R$,

$$g(x) = \sum_{j=1}^{N} \sum_{\|hv\| \leqslant R} h^{-n/p} c_j^h(v)\, \phi_j(x/h - v).$$

Since all $\phi_j \in E_k(\mathbb{R}^n)$ $(j = 1, ..., N)$, we deduce from (5.4) that

$$\begin{aligned} |g(x)| &\leqslant C \sum_{j=1}^{N} \sum_{\|hv\| \leqslant R} h^{-n/p} |c_j^h(v)|\, \|x/h - v\|^{-n-k-\lambda} \\ &\leqslant Ch^{-n/p}(\|x\|/h)^{-n-k-\lambda} \sum_{j=1}^{N} \sum_{\|hv\| \leqslant R} |c_j^h(v)|. \end{aligned}$$

Since c_j^h satisfy the condition (iii), by Hölder's inequality we have

$$\sum_{\|hv\| \leqslant R} |c_j^h(v)| \leqslant \left(\sum_{\|hv\| \leqslant R} |c_j^h(v)|^p \right)^{1/p} \left(\sum_{\|hv\| \leqslant R} 1 \right)^{1/q} \leqslant C \|u\|_p h^{-n/q}.$$

Putting the above estimates together, we obtain

$$|g(x)| \leqslant Ch^{k+\lambda} \|x\|^{-k-n-\lambda} \|u\|_p, \qquad \text{for} \quad \|x\| \geqslant 2R.$$

Note that u is a fixed function. Hence it follows from the above estimate that

$$J_2 \leqslant Ch^{k+\lambda} \int_{\|x\| \geqslant 2R} \|x\|^{m-k-n-\lambda}\, dx \leqslant Ch^k.$$

To summarize, we have proved our claim (5.2).

The Fourier transform of u can be easily computed:

$$\hat{u}(\xi) = \prod_{j=1}^{n} \left\{ \frac{\sin(\xi_j/2)}{\xi_j/2} \right\}^{k+l}, \qquad \xi = (\xi_1, ..., \xi_n) \in \mathbb{R}^n.$$

Hence

$$\hat{u}(0) = 1 \tag{5.5}$$

and

$$\lim_{h \to 0} D^\alpha \hat{u}(\xi/h)/h^{k-1} = 0 \qquad \text{for} \quad \xi \in \mathbb{R}^n \backslash \{0\} \text{ and } |\alpha| < k. \tag{5.6}$$

Recall that $g = u - u_h$. Thus (5.2), (5.5), and (5.6) together yield

$$\lim_{h \to 0} \hat{u}_h(0) = 1$$

and

$$\lim_{h \to 0} D^\alpha \hat{u}_h(\xi/h)/h^{k-1} = 0 \qquad \text{for all} \quad \xi \in \mathbb{R}^n \setminus \{0\} \text{ and } |\alpha| < k.$$

Therefore it remains to prove the following:

LEMMA 5.2. *Let k be a positive integer and Φ a finite collection of normal functions in $E_k(\mathbb{R}^n)$. For each $h > 0$, let $S_h(\Phi)$ denote the linear space spanned by $\phi(\cdot/h - v)$, where $\phi \in \Phi$, $v \in \mathbb{Z}^n$. Suppose that there is a family $(u_h)_{h>0}$ of functions satisfying the conditions*

(a) $u_h \in S_h(\Phi)$ *for each* $h > 0$,

(b) $\lim_{h \to 0} \hat{u}_h(0) = 1$, *and*

(c) $\lim_{h \to 0} D^\alpha \hat{u}_h(\xi/h)/h^{k-1} = 0$ *for* $\xi \in \mathbb{R}^n \setminus \{0\}$ *and* $|\alpha| < k$.

Then Φ satisfies the Strang–Fix conditions of order k.

The argument given in [3] can be carried over verbatim to prove Lemma 5.2. Thus the proof of Theorem 5.1 is complete. ∎

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